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Letter to the editor

Comments on the "Finite Element Solution of the Stability Problem for Nonlinear Undamped and Damped Systems Under Nonconservative Loading", *Int. J. Solids Structures* Vol. 34, No. 19, pp. 2497–2516 (1997) by R.V. Vitaliani, A.M. Gasparini and A.V. Saetta

The paper by Vitaliani et al. (1997) provides means to analyze nonlinear behavior and infinitesimal stability of structures which can be modeled as an assemblages of 3-D elastic beams. The presented finite element formulation allows to calculate the eigenvalues of loaded structure, thus defining the critical value of the load. Since the considered problems are intended to serve as the benchmark tests, the verification of the results by methods other than the finite element is desirable.

This writer has checked analytically the load-displacement diagrams for problems B5, B13, and B18, presented by Vitaliani et al. (1997). Since the authors use the displacement approximation from Surana and Sorem (1989), one numerical example from this paper was also verified analytically.

The analytical solutions presented here are based on the well known governing equations for the geometrically nonlinear plane deformation of an inextentional curved beam with the constant initial curvature [see, e.g., Love (1944), Chernykh (1986), Detinko (1998)]

$$Q' + \gamma' N = 0,\ N' - \gamma' Q = 0,\ Q = M',\ M = -\beta',\ \beta = \gamma - \varphi \tag{1a-e}$$

Here β is the rotation angle, Q, N are the shear and tension-compression stress resultants forces and M is the bending moment. The equations (1) are written in the distorted coordinate system $(n,\ t)$ and are appropriate for describing deformations under nonconservative terminal forces which remain normal (Q-direction) or tangential (N-direction) to the beam deformed center line. All forces are dimensionless and are related to the physical forces $(F_n,\ F_t,\ M_y)$ by

$$(Q,\ N) = \left(R^2/EI\right)(F_n,\ F_t),\ M = (R/EI)M_y \tag{2}$$

A prime denotes the derivative with respect to the dimensionless coordinate $\varphi = s/R$. The (x,z) components of displacement are found from

$$u_x' = \sin\gamma - \sin\varphi,\ u_z' = \cos\gamma - \cos\varphi \tag{3}$$

and the (n,t) components from

$$w = u_x \cos\gamma - u_z \sin\gamma,\ v = u_x \sin\gamma + u_z \cos\gamma \tag{4}$$

Expressing the shear force in (1b) in terms of γ and integrating the resulting relations one obtains

$$N = C - \Phi^2/2,\ \Phi = \gamma' \tag{5}$$

where C is an integration constant. Combining this with (1a) yields

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$$\Phi'' - C\Phi + \tfrac{1}{2}\Phi^3 = 0 \tag{6}$$

Thus, all variables are expressed in terms of the curvature. For the straight beam one should set $\varphi = 0$ in (1e) and (3), and replace the independent variable by $x = s/L$.

1. **Numerical example 1 in Surana and Sorem (1989).** The cantilever beam loaded by a concentrated end moment M_0 was analyzed using four three node elements. In this case of pure bending the exact solution is given by

$$M = M_0 = \text{constant},\ \beta = -M_0 x,\ Q = N = 0 \tag{7}$$

with $x = 0$ at the fixed end. From (3) the displacements are

$$u_x(x) = -\int_0^x \sin(M_0 x)\,\mathrm{d}x = \frac{\cos(M_0 x) - 1}{M_0}$$

$$u_z(x) = \int_0^x \left[\cos(M_0 x) - 1\right]\mathrm{d}x = \frac{\sin(M_0 x)}{M_0} - x$$

The beam is bent into an arc of a circle (or into a complete circle for $M_0 \geq 2\pi$) tangent to the initial center line, so that after the deformations all points are located on the same side of the undeformed center line. To obtain the numerical values (δ_x, δ_y) given in Table 1 of Surana and Sorem (1989), our dimensionless displacements were multiplied by the beam length L = 12 in. or $\delta_x = Lu_z$ (1), $\delta_y = Lu_x$ (1). Comparison of the exact and finite element results is presented in Table 1 (all variables are multiplied by -1).

For the highest load the inaccuracy of the finite element results reaches 13% for the rotation, 8% for the horizontal displacement, and 30% for the vertical displacement (the sign of this relatively low displacement is wrong). Probably the number of elements for this load was not sufficient.

2. **Problem B5 of Vitaliani et al. (1997).** For the cantilever beam under follower normal force the solution of (6) is taken as

$$-M = \Phi = M_0\frac{\mathrm{sn}(hx)}{\mathrm{dn}(hx)},\ Q = M' = -hM_0\frac{\mathrm{cn}(hx)}{\mathrm{dn}^2(hx)} \tag{8}$$

where $\mathrm{sn}(x,k)$, $\mathrm{cn}(x,k)$, $\mathrm{dn}(x,k)$ are Jacoby elliptic functions of modulus k. Substituting (8) into (6) and using relationships

$$\mathrm{sn}^2(x) + \mathrm{cn}^2(x) = 1,\ \mathrm{dn}^2(x) + k^2\mathrm{sn}^2(x) = 1$$

Table 1
Cantilever beam under concentrated moment

	Surana and Sorem (1989)			Exact		
M_0/π	β	δ_x	δ_y	β	δ_x	δ_y
0.2	0.6376	0.7652	3.623	0.6283	0.7741	3.648
0.6	1.925	5.955	8.328	1.885	5.945	8.334
1.0	3.259	12.22	7.507	3.142	12.00	7.639
1.4	4.700	14.66	2.877	4.398	14.59	3.572
1.8	6.397	12.18	−0.1232	5.655	13.25	0.4053

to express all members in terms of $\mathrm{sn}(x)$ one finds

$$C = (2k^2 - 1)h^2,\ M_0^2 = 4k^2(1 - k^2)h^2 \tag{9}$$

At the loaded end ($x = 0$) (8) satisfies the boundary condition $M(0)=0$. Conditions $N(0)=0$, $Q(0)=-Q_0$ yield $k^2=1/2$, $h^2=Q_0$. The rotation

$$\beta(x) = \int_x^1 M(x)\,\mathrm{d}x = 2\ \mathrm{arc\ tan}(\mathrm{cn}h) - 2\ \mathrm{arc\ tan}(\mathrm{cn}hx)$$

and the displacements are obtained by a numerical integration:

$$u_x(x) = \int_1^x \sin\beta\,\mathrm{d}x,\ u_z(x) = \int_1^x (\cos\beta - 1)\,\mathrm{d}x$$

To calculate the Jacoby functions we used their representation as an infinite trigonometric product, Gradstein and Ryzhic (1980).

The numerical results are presented in Table 2 (here and below R, V, H stand for the rotation and the dimensionless vertical and horizontal displacements):

By estimation from Fig. 2 of Vitaliani et al. (1997) for the load $P = 120\ kN$ the rotation is $0.39\pi = 1.22$ and from Table 1 the dimensionless displacements are -0.406, 0.652 with an error only 5%.

Problem B13 (Circular arch under normal follower force)

Substituting the assumed solution

$$\Phi = \Phi_0 \frac{\mathrm{sn}(h\varphi + \tau)}{\mathrm{dn}(h\varphi + \tau)},$$

into (6) one finds

$$C = (2k^2 - 1)h^2,\ \Phi_0^2 = 4k^2(1 - k^2)h^2$$

The boundary conditions at the loaded end

$$M(0) = 1 - \Phi(0) = 0,\ N(0) = C - \Phi^2(0)/2 = 0,\ Q(0) = -\Phi'(0) = -Q_0$$

yield three equations

Table 2
Cantilever beam under follower normal force

P, kN	R	V	H
0.7	0.1000	0.0666	0.00266
3.5	0.4959	0.3206	0.0644
14	1.776	0.7862	0.6573
56	3.099	0.4212	0.9090
84	2.444	0.1587	0.5652
120	1.162	−0.4042	0.6490
130	0.8522	−0.4786	0.7586

$$\Phi_0 \mathrm{sn}\tau = \mathrm{d}n\tau,\ 2(2k^2 - 1)h^2 = 1,\ h\Phi_0 \mathrm{cn}\tau = Q_0\ \mathrm{dn}^2\tau$$

from which after some manipulations one finds

$$h^2 = Q_0,\ k^2 = \frac{2Q_0 + 1}{4Q_0},\ \mathrm{sn}^2\tau = \frac{2}{1 + 2Q_0}$$

Using the boundary conditions

$$\beta(\alpha) = u_x(\alpha) = u_z(\alpha) = 0,\ \beta(\alpha) = u_x(\alpha) = u_y(\alpha) = 0,\ \beta(\alpha) = u_x(\alpha) = u_y(\alpha) = 0,$$

(where α is the arch subtended angle), the rotation and displacements are found by a numerical integration, as before. From the requirement $k^2 < 1$ it follows that this solution is valid for $Q_0 < 1/2$. Otherwise the solution is

$$\Phi = \Phi_0 \mathrm{dn}^{-1}(h\varphi + \tau),\ \Phi_0^2 = \mathrm{dn}^2\tau = 1 - 2Q_0,\ k^2 = \frac{4Q_0}{2Q_0 + 1},\ h^2 = (1 + 2Q_0)/4$$

Table 3 lists the numerical results.

The horizontal displacement from Table 2 of Vitaliani et al. (1997) is equal to the exact value, but small vertical motion differs by 26%.

Problem B18 (Right angled frame under follower force). For this problem the authors remark that "The results obtained…differ substantially from those published by Argyris and Symeondis (1981a)". There is a simple way to verify the starting portion of the load-displacement curve: for small displacements the results can be calculated by the linear approximation. For small displacements one easily obtains the rotation and displacements under the load $P = 1\ kN$:

$$\varphi = \frac{PL^2}{2EI} + \frac{PL^2}{EI} = 0.090,\ u_x/L = \frac{PL^2}{3EI} + \frac{PL^2}{EI} = 0.080,\ u_z/L = \frac{PL^2}{2EI} = 0.030$$

First component of the rotation and horizontal displacement is due to bending of the vertical beam, the second one is caused by the moment PL applied to the end of horizontal beam, which also causes the downward displacement of the loaded section. The displacement due to compression is negligible. The obtained deformations are low enough to justify the use of linear solution. By estimation from Fig. 7 of Vitaliani et al. (1997) for the same load $\varphi/\pi = 0.25$, $u_x/L = 0.64$. One is inclined to attribute this huge discrepancy to some typographic error.

The considered examples imply that, for the load near critical, small displacements, calculated by the developed finite element procedure, may differ substantially from the closed form solution. Otherwise

Table 3
Circular arch under follower normal force

P, kN	*R*	*V*	*H*
0.05	−0.0667	0.0631	0.0544
0.20	−0.2357	0.1861	0.2083
1	−0.5861	0.0102	0.6347
2	−0.2560	−0.8664	0.9182
3	0.5802	−1.664	1.954
4	1.608	−1.213	3.328
5	2.468	−0.1658	3.735

the results obtained by the two methods are in close agreement.

References

Chernykh, K.F., 1986. Non-linear Theory of Elasticity. Mashinostroenie, Leningrad (in Russian).

Detinko, F.M., 1998. Circular arch with no instability. Int. J. Solids Structures 35, 3213–3220.

Gradstein, I.S., Ryzhic, H.I.M., 1980. Tables of Integrals, Series, and Products. Academic Press.

Love, A.E.H., 1944. A Treatise on the Mathematical Theory of Elasticity. Dover.

Surana, K.S., Sorem, R.M., 1989. Geometrically non-linear formulation for three dimensional curved beam elements with large rotations. Int. J. Numerical Methods in Engineering 28, 43–73.

Vitaliani, R.V., Gasparini, A.M., Saetta, A.V., 1997. Finite element solution of the stability problem for nonlinear undamped and damped systems under nonconservative loading. Int. J. Solids Structures 34, 2497–2516.

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